The singular behavior of one-loop massive QCD amplitudes with one external soft gluon

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Abstract

We calculate the one-loop correction to the soft-gluon current with massive fermions. This current is process independent and controls the singular behavior of one-loop massive QCD amplitudes in the limit when one external gluon becomes soft. The result derived in this work is the last missing process-independent ingredient needed for numerical evaluation of observables with massive fermions at hadron colliders at the next-to-next-to-leading order.

1. Introduction

The main obstacle for the numerical evaluation of collider observables at higher perturbative orders is the presence of infrared (IR) (soft and collinear) divergences in parton level calculations. These divergences cancel in observables, but need to be regularized in all intermediate calculations by introducing an appropriate parameter (typically dimensionally). It is the need to keep track of such a regularization parameter that prevents the *ab initio* application of straightforward methods for numerical integration.

At the next-to-leading order (NLO), this complication can be evaded within the so-called subtraction method. Its basic idea is simple: first, one utilizes the universality and factorization property of IR singularities to construct an approximation to the corresponding real emission partonic amplitude. This approximation is simple enough to allow the analytic extraction, and eventually cancellation, of IR singularities. Second, one explores the fact that the difference of the full amplitude and its approximation is IR finite and therefore can be integrated numerically in a straightforward way. This approach, in effect, splits the task of performing a complicated divergent integration in two: first, a divergent integration of a simpler quantity and, second, a complicated, but finite numerical integration. The subtraction approach can be applied to processes with massless and massive fermions. Several subtraction schemes have been proposed [1, 2, 3, 4, 5, 6] and have been successfully used in a large number of applications.

To construct a subtraction scheme at next-to-next-to-leading order (NNLO) one needs to know, among others, the limiting behavior of one-loop amplitudes when one of the external *on-shell* partons - a gluon - becomes soft. For the case of massless fermions (like massless QED or QCD) this problem has been studied in specific cases in Refs. [7, 8] and later in Ref. [9]. The goal of the present work is to generalize the process-independent approach of Catani and Grazzini [9] to the case of massive fermions. With the result of one of the authors for the treatment of double real radiation [10, 11], the result derived in this paper represents the last missing ingredient for the construction of NNLO observables with massive fermions. Applications are top and bottom (and charm) production at hadron colliders, deep inelastic scattering and processes at lepton colliders.

This paper is organized as follows: In Section 2 we present the factorization of one-loop amplitudes in the soft limit and introduce the one-loop soft-gluon current. There we also present the derivation of the one-loop soft-gluon current in terms of a set of scalar integrals. In Sections 2.1, 2.2 and 2.3 we present the

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explicit result for the UV unrenormalized one-loop soft-gluon current for all phenomenologically relevant kinematical configurations. In Section 2.4 we discuss the properties of the results and explain a number of checks we have performed. In Section 3 we derive the soft limit of squared matrix elements as needed in specific applications. In Section 4 we describe the UV renormalization of the bare one-loop soft-gluon current followed by a summary. We have added a number of appendices containing most of the technical details. The evaluation of all scalar integrals is detailed in Appendix A. In Appendix B and Appendix C we independently derive two limiting results for the UV renormalized one-loop soft current: its small-mass limit and its pole terms, respectively. Finally, in Appendix D, we discuss the analytical continuation of the bare one-loop soft-gluon current evaluated in different kinematical configurations.

2. Amplitude factorization in the soft limit

Consider the amplitude $M_a(n+1;q)$ for producing $n+1$ on-shell partons. [1] Let at least one final state parton be a gluon, and let $a = 1,\ldots,N_c^2$ be its color index and q its momentum $q^2 = 0$. It is useful to think of $M_a(n+1;q)$ as a wide-angle scattering amplitude, i.e. all kinematical invariants formed from its external momenta are large. The structure of such amplitudes is very well understood through at least two-loops [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27] in both the massive and the massless cases. We are next interested in the limit when the external gluon becomes soft $q \to 0$, or more precisely its momentum scales as:

$$q \to \lambda q,\ \ \lambda \to 0\,. \tag{1}$$

Clearly, in the limit (1), $M_a(n+1)$ is not a wide-angle scattering amplitude anymore. Instead, it satisfies the following factorization property:

$$M_a(n+1;q) = J_a(q) M(n) + \mathcal{O}(\lambda)\,. \tag{2}$$

In the above equation, the amplitude $M(n)$ is the wide-angle scattering amplitude obtained from $M_a(n+1;q)$ by removing the external gluon with momentum q, and $J_a(q)$ is the process-independent soft-gluon (eikonal) current whose derivation is the main goal of this work. We have adopted a concise notation in Eq. (2), but have made explicit the dependence on q, a and n.

Each one of the factors in Eq. (2) depends on the dimensional regularization parameter $\epsilon = (4-d)/2$ and has a loop expansion in powers of the strong coupling constant through one loop:

$$\begin{aligned} J_a(q) &= g_S \mu^\epsilon \left(J_a^{(0)}(q) + J_a^{(1)}(q) + \ldots \right)\,, \\ M(n) &= M^{(0)}(n) + M^{(1)}(n) + \ldots\,, \\ M_a(n+1;q) &= M_a^{(0)}(n+1;q) + M_a^{(1)}(n+1;q) + \ldots\,, \end{aligned} \tag{3}$$

where the dots stand for terms at higher orders in α_S. The notation adopted in Eq. (3) does not make explicit the powers of the strong coupling α_S. For example, $J_a^{(0)}$ denotes the leading order result for the soft current (given explicitly in Eq. (4) below), $J_a^{(1)}$ stands for its next-to-leading order in α_S, and so on. The reason for choosing this notation is that the leading order amplitude $M^{(0)}(n)$ contains a process dependent power of the strong coupling constant. Thus, our notation reflects the only relevant information: the power of the strong coupling relative to the leading order amplitude $M^{(0)}(n)$.

Our considerations apply for both bare and UV renormalized amplitudes. For now we consider bare amplitudes and will return to the UV renormalization in Section 4.

Both in the massive ($p_i^2 > 0$) and massless ($p_i^2 = 0$) cases the tree level soft-gluon current reads:

$$J_a^{\mu(0)}(q) = \sum_{i=1}^{n} T_i^a \frac{p_i^\mu}{p_i \cdot q} \equiv \sum_{i=1}^{n} T_i^a e_i^\mu\,, \tag{4}$$

[1] Unless we state otherwise, we do not make a distinction between initial and final state partons.

where $J_a^{(n)}(q) \equiv \varepsilon^\mu(q) J_a^{\mu(n)}(q)$. Throughout we follow the conventions of Ref. [9] for the signs of color generators. The one-loop UV un-renormalized soft-gluon current $J_a^{\mu(1)}(q)$ reads:

$$J_a^{\mu(1)}(q) = i f_{abc} \sum_{i \neq j=1}^{n} T_i^b T_j^c \left(e_i^\mu - e_j^\mu\right) g_{ij}^{(1)}(\epsilon, q, p_i, p_j)\,. \tag{5}$$

For the calculation of the one-loop soft-gluon current in Eq. (5) we follow the strategy developed in Ref. [9]. The approach consists of the evaluation of all one-loop diagrams connecting (on-shell) external legs and attaching a real gluon to either the external legs or the gluon propagator (the virtual gluon). The calculation is performed in the eikonal approximation, i.e. the real and virtual gluons are treated as being of similar magnitude, and energy-momentum conservation is enforced.

Following the terminology introduced in Ref. [9], we split the results in $1P$ and $2P$ contributions. The $1P$ contributions are defined as the ones that depend on a single external hard momentum p_i, as opposed to the $2P$ contributions that involve two hard momenta p_i and p_j. In the following we calculate the $2P$ contributions and show that they are separately conserved; then, adapting the arguments given in Ref. [9] one can show that the $1P$ terms do not contribute to the soft current.

Our starting point for the calculation of the $2P$ contribution $J_{a(2P)}^{\mu(1)}$ to the one-loop soft-gluon current is the sum of the three diagrams $4(a,b,c)$ given in Ref. [9]. [2] We neglect all scaleless integrals. The sum of diagrams is gauge invariant as also explained in [9]. The term $\sim k \cdot \varepsilon(q)$ is reduced to scalar integrals. The reduction differs from the one in Ref. [9] since in the case at hand, it produces terms that explicitly depend on the masses $m_{i,j}^2$. Applying partial fractioning and omitting scaleless integrals, we arrive at the following expression for the function $g_{ij}^{(1)}$ in Eq. (5):

$$\begin{aligned} g_{ij}^{(1)} \quad &= \quad a_S^b \mu^{2\epsilon}\, \frac{p_i \cdot p_j}{m_i^2 (p_j \cdot q)^2 - 2(p_i \cdot p_j)(p_i \cdot q)(p_j \cdot q) + m_j^2 (p_i \cdot q)^2} \\ &\times \quad \Bigg\{ (p_i \cdot q)(p_j \cdot q) \left[(p_j \cdot q) M_1 + (p_i \cdot q) \hat{M}_1\right] \\ &\quad + \frac{1}{2}(p_j \cdot q)\left[(p_i \cdot p_j)(p_i \cdot q) - m_i^2 (p_j \cdot q)\right] M_2 + \frac{1}{2}(p_i \cdot q)\left[(p_i \cdot p_j)(p_j \cdot q) - m_j^2 (p_i \cdot q)\right] \hat{M}_2 \\ &\quad + \left[(p_i \cdot p_j)(p_i \cdot q)(p_j \cdot q) - m_i^2 (p_j \cdot q)^2 - m_j^2 (p_i \cdot q)^2\right] \frac{(p_i \cdot q)(p_j \cdot q)}{p_i \cdot p_j} M_3 \Bigg\}\,. \end{aligned} \tag{6}$$

The bare coupling $a_S^b = \alpha_s^b S_\epsilon/(2\pi)$ with $S_\epsilon = (4\pi)^\epsilon \exp(-\epsilon \gamma_E)$, $\hat{M}_k \equiv M_k(p_i \leftrightarrow p_j)$, $k = 1, 2, 3$ and the integrals $M_{1,2,3}$ can be found in Appendix A. Noting that $\hat{M}_3 = M_3$, it is apparent that $g_{ij}^{(1)} = g_{ji}^{(1)}$. From Eqs. (4,5) it is evident that the massive soft-gluon current is conserved through one loop. This follows from color conservation (as explained in Ref. [9]) and the identity $q \cdot e_i = 1$.

Next we present our main result, namely, the explicit expression for the function $g_{ij}^{(1)}$. There are three kinematical regions for which this function needs to be computed. In all three cases we consider $p_i^2 = m_i^2 > 0$ and take p_i, as well as the momentum q of the soft-gluon, to be in the final state. Thus, the three kinematical configurations are defined as:

1. $p_j^2 = 0,\ p_j$ incoming,
2. $p_j^2 = 0,\ p_j$ outgoing,
3. $p_j^2 = m_j^2 > 0,\ p_j$ outgoing.

We do not have in mind phenomenological applications with massive quarks in the initial state, but for completeness, have calculated and presented below all required ingredients for such applications as well.

[2] Since, for these diagrams, at the integrand level the eikonal approximation is identical in the massive and the massless cases, we can simply use the sum of the expressions given in Eq. (46,47) of Ref. [9]. We have verified the agreement.

2.1. Case 1

We have evaluated this kinematical configuration directly, as described above, by substituting the explicit results Eqs. (A.2, A.4, A.14) for the scalar integrals into Eq. (6) and then expanding in epsilon to the desired depth. The result for the un-renormalized one-loop soft current reads:

$$g_{ij}^{(1)}(\textit{Case 1}) = R_{ij}^{[C1]} + i\pi I_{ij}^{[C1]} \equiv a_S^b \left(\frac{2(p_i\cdot p_j)\mu^2}{2(p_i\cdot q)2(p_j\cdot q)}\right)^{\epsilon} \sum_{n=-2}^{2} \epsilon^n \left(R_{ij}^{(n)[C1]} + i\pi I_{ij}^{(n)[C1]}\right) , \tag{7}$$

The bare coupling a_S^b is introduced in Eq. (6) and:

$$\begin{aligned}
I_{ij}^{(-2)[C1]} &= 0\,, \\
I_{ij}^{(-1)[C1]} &= -\frac{1}{2}\,, \\
R_S\, I_{ij}^{(0)[C1]} &= 2m_i^2(p_j\cdot q)\ln\left(\frac{\alpha_i}{2}\right)\,, \\
R_S\, I_{ij}^{(1)[C1]} &= 4\left[(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)\right]\mathrm{Li}_2\left(1-\frac{\alpha_i}{2}\right) + m_i^2(p_j\cdot q)\ln^2\left(\frac{\alpha_i}{2}\right) \\
&\quad + \pi^2\frac{-2(p_i\cdot p_j)(p_i\cdot q) + m_i^2(p_j\cdot q)}{2}\,, \\
R_S\, I_{ij}^{(2)[C1]} &= 4\left[(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)\right]\left[\mathrm{Li}_3\left(1-\frac{\alpha_i}{2}\right) + \mathrm{Li}_3\left(\frac{\alpha_i}{2}\right)\right] - \zeta_3\frac{40(p_i\cdot p_j)(p_i\cdot q) - 26m_i^2(p_j\cdot q)}{3} \\
&\quad + 2\left[(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)\right]\ln\left(1-\frac{\alpha_i}{2}\right)\ln^2\left(\frac{\alpha_i}{2}\right) + \frac{m_i^2(p_j\cdot q)}{3}\ln^3\left(\frac{\alpha_i}{2}\right) \\
&\quad + \ln\left(\frac{\alpha_i}{2}\right)\left(\pi^2\frac{-4(p_i\cdot p_j)(p_i\cdot q) + m_i^2(p_j\cdot q)}{6} + 4\left[(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)\right]\mathrm{Li}_2\left(1-\frac{\alpha_i}{2}\right)\right) \\
R_{ij}^{(-2)[C1]} &= -\frac{1}{2}\,, \\
R_{ij}^{(-1)[C1]} &= 0\,, \\
R_S\, R_{ij}^{(0)[C1]} &= m_i^2(p_j\cdot q)\ln^2\left(\frac{\alpha_i}{2}\right) - \pi^2\frac{5(2(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q))}{6}\,, \\
R_S\, R_{ij}^{(1)[C1]} &= 4\left[(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)\right]\mathrm{Li}_3\left(\frac{\alpha_i}{2}\right) - \zeta_3\frac{4\left[7(p_i\cdot p_j)(p_i\cdot q) - 5m_i^2(p_j\cdot q)\right]}{3} \\
&\quad + 2\left[(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)\right]\ln\left(1-\frac{\alpha_i}{2}\right)\ln^2\left(\frac{\alpha_i}{2}\right) \\
&\quad + \ln\left(\frac{\alpha_i}{2}\right)\left(\pi^2\frac{-2(p_i\cdot p_j)(p_i\cdot q) - 5m_i^2(p_j\cdot q)}{3} + 4\left[(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)\right]\mathrm{Li}_2\left(1-\frac{\alpha_i}{2}\right)\right)\,, \\
R_S\, R_{ij}^{(2)[C1]} &= -4\left[(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)\right]\left[\mathrm{Li}_4\left(1-\frac{2}{\alpha_i}\right) + \mathrm{Li}_4\left(1-\frac{\alpha_i}{2}\right) - \mathrm{Li}_4\left(\frac{\alpha_i}{2}\right)\right] \\
&\quad + \pi^4\frac{458(p_i\cdot p_j)(p_i\cdot q) - 213m_i^2(p_j\cdot q)}{720} \\
&\quad + \ln\left(\frac{\alpha_i}{2}\right)\left(4\left[(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)\right]\mathrm{Li}_3\left(1-\frac{\alpha_i}{2}\right) - 2\zeta_3\left[2(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)\right]\right) \\
&\quad + \pi^2\frac{-4(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)}{12}\ln^2\left(\frac{\alpha_i}{2}\right) + 2\frac{(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)}{3}\ln\left(1-\frac{\alpha_i}{2}\right)\ln^3\left(\frac{\alpha_i}{2}\right) \\
&\quad + \frac{-2(p_i\cdot p_j)(p_i\cdot q) + 3m_i^2(p_j\cdot q)}{12}\ln^4\left(\frac{\alpha_i}{2}\right) - \pi^2\frac{14}{3}\left[(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)\right]\mathrm{Li}_2\left(1-\frac{\alpha_i}{2}\right)\,.
\end{aligned} \tag{8}$$

We find it convenient to express the result through the variables R_S and α_i defined as:

$$R_S = 4\left[m_i^2(p_j\cdot q) - 2(p_i\cdot p_j)(p_i\cdot q)\right]\ ,\ \alpha_i = \frac{m_i^2(p_j\cdot q)}{(p_i\cdot q)(p_i\cdot p_j)}\ ,\ \alpha_j = \frac{m_j^2(p_i\cdot q)}{(p_j\cdot q)(p_i\cdot p_j)}\,. \tag{9}$$

We introduce the variable α_j for later use. The result in Eq. (8) is also available in electronic form.

2.2. *Case 2*

To derive this case we have performed an analytical continuation of the result from *Case* 1. When $p_j^2 = 0$ the continuation amounts to exchanging $p_j \to -p_j$ (see Appendix D). It is easy to see that Eq. (8) remains unchanged under this transformation, i.e. the result in *Case* 2 is identical to that in *Case* 1.

2.3. *Case 3*

We calculate the result for *Case 3* up to and including terms of $\mathcal{O}(\epsilon)$. We note that the $\mathcal{O}(\epsilon^2)$ term contributes only, if multiplied by a term $\sim 1/\epsilon^2$ originating from the phase-space integration. Such a leading pole may only be due to the emission of soft and collinear radiation. Both partons i and j beeing massive, collinear singularities are regularized and do not lead to poles in ϵ.

We compute the result for the one-loop soft-gluon current in this kinematical configuration directly, as expansion in ϵ. Details of the calculation can be found in Appendix A. For completeness, and with more formal applications in mind, we have also calculated all integrals in the spacelike region, where the massive momentum p_j is incoming; see Appendix A. The relation between the results in the two kinematical configurations is discussed in Appendix D.

The explicit result for the soft-gluon current in the kinematics of *Case 3* reads:

$$g_{ij}^{(1)}(\textit{Case 3}) = R_{ij}^{[C3]} + i\pi I_{ij}^{[C3]} \equiv \; a_S^b \left(\frac{2(p_i\cdot p_j)\mu^2}{2(p_i\cdot q)2(p_j\cdot q)}\right)^{\epsilon} \sum_{n=-2}^{1} \epsilon^n \left(R_{ij}^{(n)[C3]} + i\pi I_{ij}^{(n)[C3]}\right) . \tag{10}$$

and:

$$\begin{aligned}
I_{ij}^{(-2)[C3]} &= 0\,, \\
I_{ij}^{(-1)[C3]} &= -1 + \frac{1}{2v}\,, \\
I_{ij}^{(0)[C3]} &= \frac{\ln(v)}{v} + \frac{\ln(x)}{2v} + \left(1+\frac{1}{2v}\right)\ln(1+x^2) \\
&\quad + \frac{1}{Q_S}\left[-4\frac{m_j^2(p_i\cdot q)^2 - m_i^2(p_j\cdot q)^2}{v}\ln\left(\frac{\alpha_i}{\alpha_j}\right) + 16(p_i\cdot p_j)(p_i\cdot q)(p_j\cdot q)\ln(x)\right], \\
I_{ij}^{(1)[C3]} &= \frac{1}{v}\left(\frac{1}{16}\ln^2\left(\frac{\alpha_i}{\alpha_j}\right) + \mathrm{Li}_2\left(x^2\right) + \ln(v)\left(\ln\left(x^2+1\right)+\ln(x)\right) + \ln^2(v) + \frac{1}{4}\ln^2\left(x^2+1\right)\right. \\
&\quad \left. + \frac{1}{2}\ln(x)\ln\left(x^2+1\right) + \frac{\ln^2(x)}{4} - \frac{\pi^2}{8}\right) \\
&\quad + \frac{1}{Q_S}\Big[(p_i\cdot p_j)(p_i\cdot q)(p_j\cdot q)\left(2\ln^2\left(\frac{\alpha_i}{\alpha_j}\right) - 16\ln\left(x^2+1\right)\ln(x) + 8\ln^2(x) - \frac{8\pi^2}{3}\right) \\
&\quad + \left(m_i^2(p_j\cdot q)^2 + m_j^2(p_i\cdot q)^2\right)\left(8\ln^2\left(x^2+1\right) - \frac{4\pi^2}{3}\right) \\
&\quad -4\left(m_j^2(p_i\cdot q)^2 - m_i^2(p_j\cdot q)^2\right)\frac{1}{v}\left(2\ln(v) + \ln\left(x^2+1\right) + \ln(x)\right)\ln\left(\frac{\alpha_i}{\alpha_j}\right) \\
&\quad + \left(m_j^2(p_i\cdot q)^2 + m_i^2(p_j\cdot q)^2 - (p_i\cdot p_j)(p_i\cdot q)(p_j\cdot q)\right)\Big(\\
&\quad 32\ln(2)\left(-\ln(\alpha_i+v+1) - \ln(\alpha_j+v+1) - 2\ln\left(x^2+1\right) - \ln(x)\right) + 64\ln^2(2) \\
&\quad +16\ln\left(x^2+1\right)\left(\ln(\alpha_i+v+1)+\ln(\alpha_j+v+1)\right) + 16\ln(\alpha_i+v+1)\ln(\alpha_j+v+1) \\
&\quad +16\ln(x)\left(\ln(-\alpha_j+v+1) + \ln(\alpha_j+v-1) + 2\ln\left(x^2+1\right)\right) - 16\ln^2(x)
\end{aligned} \tag{11}$$

$$+8\ln\left(\frac{\alpha_i}{\alpha_j}\right)\left(\ln(\alpha_j+v-1)-\ln(-\alpha_j+v+1)\right)+4\ln^2\left(\frac{\alpha_i}{\alpha_j}\right)$$
$$+16\mathrm{Li}_2\left(\frac{-v+\alpha_j+1}{2\alpha_j}\right)+16\mathrm{Li}_2\left(2-\frac{2\alpha_j}{-v+\alpha_j+1}\right)-16\mathrm{Li}_2\left(\frac{-v+\alpha_j+1}{v+\alpha_j+1}\right)$$
$$+16\mathrm{Li}_2\left(\frac{v+\alpha_j+1}{2v+2}\right)-16\mathrm{Li}_2\left(-\frac{(v-1)(v+\alpha_j+1)}{(v+1)(-v+\alpha_j+1)}\right)+16\mathrm{Li}_2\left(\frac{2\alpha_j}{v+\alpha_j+1}\right)\bigg)\bigg]\,.$$

$$R_{ij}^{(-2)[C3]} = -\frac{1}{2}\,,$$
$$R_{ij}^{(-1)[C3]} = \frac{1}{2}\left(-1+\frac{1}{v}\right)\ln(x)+\frac{1}{2}\ln(1+x^2)\,,$$
$$R_{ij}^{(0)[C3]} = \frac{1}{2v}\mathrm{Li}_2(x^2)+\pi^2\left(\frac{19}{24}-\frac{7}{12v}\right)+\frac{1}{v}\ln(v)\ln(x)+\frac{1}{2}\left(1+\frac{1}{v}\right)\ln(x)\ln(1+x^2)$$
$$-\frac{1}{4}\ln^2(1+x^2)+\frac{1}{Q_S}\left[\left(m_j^2(p_i\cdot q)^2+m_i^2(p_j\cdot q)^2\right)\ln^2\left(\frac{\alpha_i}{\alpha_j}\right)\right.$$
$$\left.+4\left(m_j^2(p_i\cdot q)^2+m_i^2(p_j\cdot q)^2\right)\ln^2(x)-4\frac{m_j^2(p_i\cdot q)^2-m_i^2(p_j\cdot q)^2}{v}\ln\left(\frac{\alpha_i}{\alpha_j}\right)\ln(x)\right]\,.$$
$$R_{ij}^{(1)[C3]} = \frac{1}{v}\bigg(-\ln(v)\left(\ln(x)\ln\left(x^2+1\right)+\pi^2\right)+\frac{\ln^3(x)}{12}+\frac{\zeta(3)}{2}$$
$$+\ln(x)\left(\frac{1}{16}\ln^2\left(\frac{\alpha_i}{\alpha_j}\right)+\frac{\mathrm{Li}_2\left(x^2\right)}{2}-\frac{3}{4}\ln^2\left(x^2+1\right)-\frac{5\pi^2}{24}\right)$$
$$-\left(\frac{\mathrm{Li}_2\left(x^2\right)}{2}+\frac{5\pi^2}{12}\right)\ln\left(x^2+1\right)-\frac{1}{2}\left(2\mathrm{Li}_3\left(1-x^2\right)+\mathrm{Li}_3\left(x^2\right)\right)\bigg)$$
$$+\frac{1}{Q_S}\Big[(p_i\cdot p_j)(p_i\cdot q)(p_j\cdot q)\Big(\frac{32\ln^3(x)}{3}-\frac{280\zeta(3)}{3}-32\ln\left(x^2+1\right)\ln^2(x)$$
$$+\ln\left(x^2+1\right)\left(36\pi^2-4\ln^2\left(\frac{\alpha_i}{\alpha_j}\right)\right)+\left(48\ln^2\left(x^2+1\right)-\frac{40\pi^2}{3}\right)\ln(x)$$
$$-\frac{88}{3}\ln^3\left(x^2+1\right)\Big)$$
$$(m_j^2(p_i\cdot q)^2+m_i^2(p_j\cdot q)^2)\Big(\left(3\ln\left(x^2+1\right)+\ln(x)\right)\ln^2\left(\frac{\alpha_i}{\alpha_j}\right)$$
$$+28\ln^3\left(x^2+1\right)-44\ln^2\left(x^2+1\right)\ln(x)-\frac{70}{3}\pi^2\ln\left(x^2+1\right)+28\ln\left(x^2+1\right)\ln^2(x)$$
$$-\frac{28}{3}\ln^3(x)+\frac{2}{3}\pi^2\ln(x)+\frac{224\zeta(3)}{3}\Big)$$
$$-\frac{(m_j^2(p_i\cdot q)^2-m_i^2(p_j\cdot q)^2)}{v}\ln\left(\frac{\alpha_i}{\alpha_j}\right)\left(4\mathrm{Li}_2\left(x^2\right)+4\ln\left(x^2+1\right)\ln(x)+8\ln(v)\ln(x)-\frac{14\pi^2}{3}\right)$$
$$+(m_i^2(p_j\cdot q)^2+m_j^2(p_i\cdot q)^2-(p_i\cdot p_j)(p_i\cdot q)(p_j\cdot q))\Big(\ln^3\left(\frac{\alpha_i}{\alpha_j}\right)-\ln^2\left(\frac{\alpha_i}{\alpha_j}\right)2\ln(v)$$
$$+\ln^2\left(\frac{\alpha_i}{\alpha_j}\right)2(\ln(\alpha_i+v+1)+\ln(-\alpha_j+v+1)+\ln(\alpha_j+v-1)-3\ln(2))$$
$$+\ln\left(\frac{\alpha_i}{\alpha_j}\right)\Big(2\ln^2(v)-12\ln^2(\alpha_i+v+1)+6\ln^2(\alpha_j+v+1)-4\ln^2(x)-12\ln^2\left(x^2+1\right)$$

$$
\begin{aligned}
&+28\ln(2)\ln(\alpha_i+v+1)+4\ln(v)(\ln(\alpha_i+v+1)-2\ln(\alpha_j+v+1)+\ln(2))-10\ln^2(2)\\
&-4\ln(\alpha_i+v+1)\ln(\alpha_j+v+1)-8\ln(2)\ln(\alpha_j+v+1)\\
&+8(2\ln(\alpha_i+v+1)-\ln(-\alpha_j+v+1)+\ln(\alpha_j+v-1)+\ln(\alpha_j+v+1)-3\ln(2))\ln(x)\\
&+24(\ln(2)-\ln(\alpha_i+v+1))\ln\left(x^2+1\right)+24\ln(x)\ln\left(x^2+1\right)-\frac{2}{3}\pi^2\Big)\\
&+\frac{32}{3}\ln^3(\alpha_i+v+1)+12\ln^3(\alpha_j+v+1)-32\ln(2)\ln^2(\alpha_i+v+1)-8\ln(v)\ln^2(\alpha_j+v+1)\\
&+4\ln(\alpha_i+v+1)\ln^2(\alpha_j+v+1)-40\ln(2)\ln^2(\alpha_j+v+1)-8\ln(v)\ln^2(x)-\frac{4}{3}\pi^2\ln(v)\\
&+8(3\ln(\alpha_i+v+1)+\ln(-\alpha_j+v+1)+\ln(\alpha_j+v-1)-5\ln(2))\ln^2(x)\\
&+40(\ln(\alpha_i+v+1)+\ln(\alpha_j+v+1)-2\ln(2))\ln^2\left(x^2+1\right)+36\ln^2(2)\ln(\alpha_i+v+1)\\
&+8\ln(v)\ln(\alpha_i+v+1)\ln\left(\frac{1}{2}(\alpha_j+v+1)\right)-4\ln^2(v)(\ln(\alpha_i+v+1)-\ln(\alpha_j+v+1))\\
&+\frac{4}{3}\pi^2(-3\ln(\alpha_i+v+1)-4\ln(\alpha_j+v+1)+7\ln(2))+8\ln(2)\ln(v)\ln(\alpha_j+v+1)\\
&-8\ln(2)\ln(\alpha_i+v+1)\ln(\alpha_j+v+1)+44\ln^2(2)\ln(\alpha_j+v+1)+4\ln^2(v)\ln(x)\\
&-24\ln^2(\alpha_i+v+1)\ln(x)-4\ln^2(\alpha_j+v+1)\ln(x)+8\ln(v)(\ln(2)-\ln(\alpha_i+v+1))\ln(x)\\
&+56\ln(2)\ln(\alpha_i+v+1)\ln(x)-8\ln(\alpha_i+v+1)\ln(\alpha_j+v+1)\ln(x)-36\ln^2(2)\ln(x)\\
&+16\ln(2)\ln(\alpha_j+v+1)\ln(x)+32\ln^2(\alpha_i+v+1)\ln\left(x^2+1\right)+80\ln^2(2)\ln\left(x^2+1\right)\\
&+32\ln^2(\alpha_j+v+1)\ln\left(x^2+1\right)-80\ln(2)\ln(\alpha_i+v+1)\ln\left(x^2+1\right)\\
&+16\ln(\alpha_i+v+1)\ln(\alpha_j+v+1)\ln\left(x^2+1\right)-80\ln(2)\ln(\alpha_j+v+1)\ln\left(x^2+1\right)\\
&+16(-4\ln(\alpha_i+v+1)-\ln(\alpha_j+v+1)+5\ln(2))\ln(x)\ln\left(x^2+1\right)-\frac{80\ln^3(2)}{3}\\
&+\left(8\ln\left(\frac{\alpha_i}{\alpha_j}\right)+16\ln(x)\right)\mathrm{Li}_2\left(\frac{1-v}{\alpha_j}\right)+\left(16\ln(x)-8\ln\left(\frac{\alpha_i}{\alpha_j}\right)\right)\mathrm{Li}_2\left(\frac{\alpha_j}{v+1}\right)\\
&+\left(4\ln\left(\frac{\alpha_i}{\alpha_j}\right)-8\ln(\alpha_i+v+1)-8\ln(\alpha_j+v+1)+24\ln(x)-16\ln\left(x^2+1\right)+4\ln^4(2)\right)\\
&\quad\times\left(\mathrm{Li}_2\left(\frac{v-1}{\alpha_j}\right)-\mathrm{Li}_2\left(\frac{\alpha_j}{\alpha_j-v+1}\right)\right)\\
&+\left(4\ln\left(\frac{\alpha_i}{\alpha_j}\right)-8\ln(\alpha_i+v+1)-8\ln(\alpha_j+v+1)-8\ln(x)-16\ln\left(x^2+1\right)+4\ln^4(2)\right)\\
&\quad\times\left(\mathrm{Li}_2\left(-\frac{v+1}{\alpha_j}\right)-\mathrm{Li}_2\left(\frac{\alpha_j}{\alpha_j+v+1}\right)\right)\\
&+8(\ln(v)-\ln(\alpha_j+v+1)+\ln(2))\mathrm{Li}_2\left(-\frac{(v-1)(\alpha_j+v+1)}{(\alpha_j-v+1)(v+1)}\right)-16\ln(x)\mathrm{Li}_2\left(x^2\right)\\
&-16\mathrm{Li}_3\left(\frac{1-v}{\alpha_j}\right)-16\mathrm{Li}_3\left(\frac{\alpha_j}{\alpha_j-v+1}\right)+8\mathrm{Li}_3\left(\frac{\alpha_j}{v-1}\right)-16\mathrm{Li}_3\left(\frac{v-1}{\alpha_j}\right)+8\mathrm{Li}_3\left(-\frac{\alpha_j}{v+1}\right)\\
&-8\mathrm{Li}_3\left(-\frac{2v}{\alpha_j-v+1}\right)-16\mathrm{Li}_3\left(\frac{\alpha_j}{v+1}\right)+8\mathrm{Li}_3\left(x^2\right)-16\mathrm{Li}_3\left(-\frac{v+1}{\alpha_j}\right)\\
&-16\mathrm{Li}_3\left(\frac{v-1}{-\alpha_j+v-1}\right)-16\mathrm{Li}_3\left(\frac{\alpha_j}{\alpha_j+v+1}\right)-8\mathrm{Li}_3\left(\frac{2v}{\alpha_j+v+1}\right)\\
&-8\mathrm{Li}_3\left(\frac{2\alpha_j v}{(v-1)(\alpha_j+v+1)}\right)-16\mathrm{Li}_3\left(\frac{v+1}{\alpha_j+v+1}\right)+8F_c\left(\frac{\alpha_j}{\alpha_j-v+1},\frac{\alpha_j}{\alpha_j+v+1}\right)\Big)\Big].
\end{aligned}
$$

The polynomial Q_S, the "conformal" variable x and relative velocity of the quark pair v read:

$$
\begin{aligned}
Q_S &= 16\left(m_j^2(p_i\cdot q)^2 - 2(p_i\cdot p_j)(p_i\cdot q)(p_j\cdot q) + m_i^2(p_j\cdot q)^2\right)\,,\\
x &= \sqrt{(1-v)/(1+v)}\,,\\
v &= \sqrt{1-\frac{m_i^2 m_j^2}{(p_i\cdot p_j)^2}}\,.
\end{aligned}
\tag{12}
$$

Beyond order $\mathcal{O}(\epsilon^0)$, the result for the one-loop soft current cannot be expressed in terms of standard polylogarithms. Multiple polylogarithms appear as evident from Eq. (A.13). At order $\mathcal{O}(\epsilon^1)$ we have combined all functions that are outside the class of the standard polylogarithms into the function:

$$
F_c(x_1,x_2) = \int_0^1 dt \frac{\ln(1-t)\ln\left(1-t\frac{x_2}{x_1}\right)}{\frac{1}{x_2}-t}\,,
\tag{13}
$$

which can be expressed in terms of multiple polylogarithms of weight 3, see Eq. (B.22) in Ref. [28]. The result Eq. (11) is also available in electronic form.

2.4. Properties and checks

The purpose of the overall d-dimensional prefactor in Eqns. (7,10) is to extract exactly the leading power scaling behavior of the one-loop soft-gluon current in the limit $q \to 0$. The remainder is given as expansion in ϵ, which has a well defined limit $q \to 0$. This is easy to see since it is invariant under independent rescaling of the momenta q, p_i, p_j.

The result for the one-loop soft-gluon current satisfies a number of consistency checks. Eq. (10) has a well defined limit, when either one of the masses m_i or m_j vanishes. In the limit $m_j \to 0$, it agrees with the result for the soft current in *Case* 2, as it should. Note that this agreement is a non-trivial check on the analytical continuation used to derive the result in *Case* 2 from that in *Case* 1. We have numerically checked the result for the hardest integral M_3 in the "time-like" kinematics *Case* 3 (see Appendix A).

We have verified that the soft current has the correct behavior in the small mass limit (see Appendix B for details). The massless limit $m_i = 0,\ m_j = 0$ of the one-loop un-renormalized soft current is regular, and the results for the soft current in all kinematical regions reproduce the massless results of Ref. [9].

We have also verified that the pole terms of the one-loop soft current agree with what is expected based on the structure of the singularities of massive gauge theory amplitudes (see Appendix C).

3. Squared matrix elements

The knowledge of the soft-gluon current makes it possible to construct an approximation to the squared one-loop matrix element for any process in the limit (1). As indicated in Eq. (2), this approximation is correct up to power suppressed terms. The result (4) for the tree-level current is exact in ϵ. The one-loop current (5) is calculated as an expansion in ϵ which is deep enough to allow the derivation of the terms $\mathcal{O}(\epsilon^0)$ in any observable at NNLO.

In the limit (1) the square of a Born amplitude reads:

$$
\begin{aligned}
&\langle M_a^{(0)}(n+1;q)|M_a^{(0)}(n+1;q)\rangle = \\
&-4\pi\alpha_S\mu^{2\epsilon}\left\{\sum_{i\neq j=1}^{n} e_{ij}\langle M^{(0)}(n)|T_i\cdot T_j|M^{(0)}(n)\rangle \;+\; \sum_{i=1}^{n}\mathcal{C}_i e_{ii}\langle M^{(0)}(n)|M^{(0)}(n)\rangle\right\} + \mathcal{O}(\lambda)\,.
\end{aligned}
\tag{14}
$$

Above we introduced $e_{ij} \equiv e_i \cdot e_j$ and $\mathcal{C}_i \equiv T_i \cdot T_i$ is the quadratic Casimir appropriate for the parton i.

The interference term between the Born and one-loop amplitude in the limit (1) reads:

$$\langle M_a^{(0)}(n+1;q)|M_a^{(1)}(n+1;q)\rangle + c.c. = -4\pi\alpha_S\mu^{2\epsilon}\Bigg\{$$
$$2C_A\sum_{i\neq j=1}^{n}(e_{ij}-e_{ii})\,R_{ij}\langle M^{(0)}(n)|T_i\cdot T_j|M^{(0)}(n)\rangle - 4\pi\sum_{i\neq j\neq k=1}^{n} e_{ik}I_{ij}\langle M^{(0)}(n)|f^{abc}T_i^aT_j^bT_k^c|M^{(0)}(n)\rangle$$
$$+\left(\sum_{i\neq j=1}^{n} e_{ij}\langle M^{(0)}(n)|T_i\cdot T_j|M^{(1)}(n)\rangle + c.c.\right) + \left(\sum_{i=1}^{n}\mathcal{C}_i e_{ii}\langle M^{(0)}(n)|M^{(1)}(n)\rangle + c.c.\right)\Bigg\} + \mathcal{O}(\lambda)\,, \quad (15)$$

where we have split $g_{ij}^{(1)} \equiv R_{ij} + i\pi I_{ij}$ into its real and imaginary parts to be found in Eqns. (7,10).

4. UV renormalization

Up to here we considered bare amplitudes. In practical applications one works with UV renormalized amplitudes. The UV renormalized one-loop soft-gluon current is very easy to derive. One needs to recognize that though that loop order no mass renormalization enters. Therefore, all one needs to do is coupling and field renormalization. Since we consistently set to zero scaleless integrals, the only correction one needs to take into account is self-energy contribution in the soft-gluon leg due to the massive flavors. It will be most convenient to work in a scheme where all massive flavors are decoupled, i.e. the coupling is running with n_f light flavors only. Then the heavy quark loop contributions into the external gluon leg will be canceled by the decoupling correction. Therefore, in order to obtain the UV renormalized current from the bare one, one only needs to perform coupling renormalization $\alpha_S^b S_\epsilon = \alpha_S(1-\beta_0\alpha_s/(2\pi\epsilon)+\mathcal{O}(\alpha_S^2))$ in the first line of Eq.(3). This procedure amounts simply to adding the term $\sim \beta_0$ in Eq. (C.5) to the bare current; see also the discussion following Eq. (C.5).

5. Summary

In this paper, we have studied the behavior of one-loop QCD amplitudes with an arbitrary number of massive fermions in the limit when one external gluon becomes soft. Similarly to the well known massless case, we find that in the limit (1), any amplitude factorizes, up to power suppressed terms, into a product of a simpler amplitude and a process independent function: the soft-gluon current.

We have explicitly calculated this current through one loop. This result enters the evaluation of any cross-section with massive fermions at next-to-next-to-leading order within a subtraction approach. An immediate application for this result is the calculation of the $t\bar{t}$ cross-section at the second perturbative order.

We have performed a number of non-trivial checks on our results. We have verified that they correctly reproduce their small-mass limit and pole terms, that we independently predict. We have performed a number of numerical checks on the non-trivial integrals.

The explicit result for the one-loop soft-gluon current with massive fermions is much more complicated than in the massless case. While it is possible to derive an exact result valid in d-dimensions, we have explicitly presented the final result in a form suitable for practical applications: we have extracted the current's leading behavior exactly in d-dimensions, and expanded the rest in ϵ in a form appropriate for calculating observables at next-to-next-to leading order. The functional form of the result is significantly more complicated and involves multiple (Goncharov) polylogarithms.

As a by-product of our calculations, and as an additional crosscheck, we have worked out the analytical continuation from space-like to time-like kinematics for a multiscale problem. In higher orders in ϵ this procedure involves multiple polylogarithms and opens an interesting subject for further investigation.

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Appendix A. The scalar integrals

The one-loop soft-gluon current can be expressed through the following integrals:

$$\begin{aligned} M_1 &\equiv \Phi \int \frac{d^d k}{i(2\pi)^d} \frac{1}{[k^2][(k+q)^2][-p_j \cdot k]} \\ M_2 &\equiv \Phi \int \frac{d^d k}{i(2\pi)^d} \frac{1}{[k^2][p_i \cdot k + p_i \cdot q][-p_j \cdot k]} \\ M_3 &\equiv \Phi \int \frac{d^d k}{i(2\pi)^d} \frac{1}{[k^2][(k+q)^2][p_i \cdot k + p_i \cdot q][-p_j \cdot k]}, \end{aligned} \tag{A.1}$$

where each propagator has an implicit $+i\delta$ imaginary part. The momenta p_i, p_j can be massive or massless and the momentum q, corresponding to the soft-gluon, is assumed outgoing and massless. The normalization factor is $\Phi = 8\pi^2(4\pi)^{-\epsilon} e^{\epsilon\gamma_E}$.

The simplest integral to evaluate is M_1:

$$M_1 = \Phi \, \frac{\pi^{-2+\epsilon}}{16} \Gamma(-\epsilon)\Gamma(2\epsilon) \frac{m_j^{2\epsilon}}{[-(p_j \cdot q) - i\delta]^{1+2\epsilon}} \,. \tag{A.2}$$

Next we consider the integral M_2. Its full q dependence can be extracted, and the remainder expressed through a one-dimensional integral:

$$\begin{aligned} M_2 &= -\Phi \, \frac{\pi^{-2+\epsilon}}{4} \Gamma(1-\epsilon)\Gamma(2\epsilon) \left[-(p_i \cdot q) - i\delta\right]^{-2\epsilon} \\ &\times \int_0^1 dt \, t^{-2\epsilon} \left\{ t^2 m_i^2 + (1-t)^2 m_j^2 - 2t(1-t)(p_i \cdot p_j) - i\delta \right\}^{-1+\epsilon} . \end{aligned} \tag{A.3}$$

This one-dimensional integral can be evaluated in terms of ${}_2F_1$-type hypergeometric functions. After some rearrangements and using standard relations between the hypergeometric functions we obtain:

$$\begin{aligned} M_2 &= \Phi \, \frac{\pi^{-2+\epsilon}}{4} \Gamma(-\epsilon)\Gamma(2\epsilon) \left[-(p_i \cdot q) - i\delta\right]^{-2\epsilon} \left[-2(p_i \cdot p_j) - i\delta\right]^{-1+\epsilon} \\ &\times \left\{ \frac{\Gamma(1+\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \, v^{-1+2\epsilon} \, \beta_j^{-\epsilon} - \frac{2\,\beta_i^{\epsilon}}{1+v} \, {}_2F_1\left(1, 1-\epsilon, 1+\epsilon; \frac{1-v}{1+v}\right) \right\} . \end{aligned} \tag{A.4}$$

We have introduced $\beta_k \equiv m_k^2/(-2(p_i \cdot p_j) - i\delta), \; k = i, j$, and the relative velocity v defined in Eq. (12). The hypergeometric function can be expanded in series in ϵ to any desired depth with the help of [29].

The most complicated integral is M_3. We first apply Schwinger α-parameterization. The two integrations, corresponding to the two propagators quadratic in k, can be transformed in the usual way:

$$\int_0^\infty d\hat{\alpha}_1 d\hat{\alpha}_2 = \int_0^\infty a da \int_0^1 dy \,, \tag{A.5}$$

and the integration over a performed.

In order to extract the scaling behavior of the integral in the limit $q \to 0$, we rescale the α-parameters $\hat{\alpha}_{3,4}$ corresponding to the two propagators that are linear in k:

$$\hat{\alpha}_3 \to \hat{\alpha}_3/|p_i \cdot q| \quad , \quad \hat{\alpha}_4 \to \hat{\alpha}_4/|p_j \cdot q| \,. \tag{A.6}$$

We note that the invariants $(p_i \cdot q)$ and $(p_i \cdot p_j)$ are non-zero, although their signs change depending on the kinematical configuration. Next we change the variables $\hat{\alpha}_{3,4}$ along the lines of Eq. (A.5) and perform the integration over the infinite range, arriving at the following two-dimensional representation for M_3:

$$\begin{aligned} M_3 &= \Phi\, \frac{\pi^{-2+\epsilon}}{16}\Gamma(-\epsilon)\Gamma(2+2\epsilon)\frac{1}{|p_i\cdot q||p_j\cdot q|}\int_0^1 dt \int_0^1 dy \left(t^2\frac{m_i^2}{(p_i\cdot q)^2} + (1-t)^2\frac{m_j^2}{(p_j\cdot q)^2} \right. \\ & \left. -2t(1-t)\frac{(p_i\cdot p_j)}{|p_i\cdot q||p_j\cdot q|} - i\delta \right)^{\epsilon} (-y\sigma_j - t\sigma_i + (\sigma_i+\sigma_j)ty - i\delta)^{-2-2\epsilon}\,, \end{aligned} \tag{A.7}$$

where $\sigma_k \equiv (p_k\cdot q)/|p_k\cdot q| = \pm 1$. Note that the signs $\sigma_{i,j}$ also determine the sign of $(p_i\cdot p_j)$.

The evaluation of Eq. (A.7) is hardest in the phenomenologically relevant "time-like" (TL) kinematics of *Case 3*, where $\sigma_i = \sigma_j = 1$. Performing the y-integration we get:

$$\begin{aligned} M_3^{(TL)} &= \Phi\, \frac{\pi^{-2+\epsilon}}{16(p_i\cdot p_j)}\Gamma(-\epsilon)\Gamma(1+2\epsilon)\,(-1-i\delta)^{-2\epsilon}\left(\frac{p_i\cdot p_j}{(p_i\cdot q)(p_j\cdot q)}\right)^{1+\epsilon}\int_0^1 dt\left(t^2\frac{m_i^2(p_j\cdot q)}{(p_i\cdot q)(p_i\cdot p_j)}\right. \\ & \left. +(1-t)^2\frac{m_j^2(p_i\cdot q)}{(p_j\cdot q)(p_i\cdot p_j)} - 2t(1-t) - i\delta\right)^{\epsilon}\left[\frac{t^{-1-2\epsilon}-(1-t)^{-1-2\epsilon}}{1-2t}\right]. \end{aligned} \tag{A.8}$$

We evaluate the above integral as expansion in ϵ using the formula:

$$z^{-1+\epsilon} = \frac{1}{\epsilon}\delta(z) + \sum_{k=0}^{\infty}\frac{\epsilon^n}{k!}\left(\frac{\ln(z)}{z}\right)_+ .$$

Extracting the imaginary parts is particularly laborious since both roots $x_{1,2}^t$ of the polynomial, which is quadratic in t, are inside the integration range: $0 \le x_2^t \le 1/2 \le x_1^t \le 1$. The expressions for the two roots in the time-like case read:

$$x_1^t = \frac{\alpha_j}{\alpha_j + 1 - v}\ ,\ x_2^t = \frac{\alpha_j}{\alpha_j+1+v}\,, \tag{A.9}$$

with α_i, α_j defined in Eq. (9). The explicit result for the integral $M_3^{(TL)}$ is rather lengthy. It is supplied in electronic form.

The integral M_3 in Eq. (A.7) is much easier to calculate in the "space-like" (SL) kinematics $\sigma_i = -\sigma_j = 1$; in the following we present its derivation for completeness. Performing the y-integration and after some simplifications we obtain:

$$\begin{aligned} M_3^{(SL)} &= \Phi\, \frac{\pi^{-2+\epsilon}}{16(p_i\cdot p_j)}\Gamma(-\epsilon)\Gamma(1+2\epsilon)\left(\frac{p_i\cdot p_j}{(p_i\cdot q)(p_j\cdot q)}\right)^{1+\epsilon}\int_0^1 dt\left(t^2\frac{m_i^2(p_j\cdot q)}{(p_i\cdot q)(p_i\cdot p_j)}\right. \\ & \left. +(1-t)^2\frac{m_j^2(p_i\cdot q)}{(p_j\cdot q)(p_i\cdot p_j)} + 2t(1-t)\right)^{\epsilon}\left[(1-t)^{-1-2\epsilon} + (-1-i\delta)^{-2\epsilon}t^{-1-2\epsilon}\right]. \end{aligned} \tag{A.10}$$

The one-dimensional integral can be evaluated in terms of the Appell hypergeometric function F_1:

$$\begin{aligned} M_3^{(SL)} &= -\Phi\, \frac{\pi^{-2+\epsilon}}{16}\frac{\Gamma(-\epsilon)\Gamma(2\epsilon)}{(p_i\cdot q)(p_j\cdot q)}\left(\frac{p_i\cdot p_j}{(p_i\cdot q)(p_j\cdot q)}\right)^{\epsilon}\Bigg\{ \\ & (-1-i\delta)^{-2\epsilon}\alpha_j^{\epsilon}F_1\left(-2\epsilon,-\epsilon,-\epsilon,1-2\epsilon;\frac{1}{x_1^s},\frac{1}{x_2^s}\right) + \alpha_i^{\epsilon}F_1\left(-2\epsilon,-\epsilon,-\epsilon,1-2\epsilon;\frac{1}{1-x_2^s},\frac{1}{1-x_1^s}\right)\Bigg\}. \end{aligned} \tag{A.11}$$

In the above equation we have introduced the following notation:

$$x_1^s = \frac{\alpha_j}{\alpha_j-1-v}\ ,\ x_2^s = \frac{\alpha_j}{\alpha_j-1+v}\,. \tag{A.12}$$

The quantities $x^s_{1,2}$ are the two roots of the polynomial quadratic in t appearing in Eq. (A.10) with $x^s_1 < 0$ and $x^s_2 > 1$ and α_i, α_j are defined in Eq. (9).

The Appell functions can be expanded in ϵ in terms of multiple polylogarithms $\mathrm{Li}_{m_k,\ldots,m_1}(t_k,\ldots,t_1)$ with the help of the library *Nestedsums* [30] (see also [31]):

$$
\begin{aligned}
&F_1(-2\epsilon,-\epsilon,-\epsilon,1-2\epsilon;t,y) = 1+\epsilon^2\left[2\mathrm{Li}_2(t)+2\mathrm{Li}_2(y)\right] \\
&+\epsilon^3\left[4\mathrm{Li}_3(t)+4\mathrm{Li}_3(y)-2S_{1,2}(t)-2S_{1,2}(y)-2\mathrm{Li}_{1,2}\left(\frac{t}{y},y\right)-2\mathrm{Li}_{1,2}\left(\frac{y}{t},t\right)\right] \\
&+\epsilon^4\left[8\mathrm{Li}_4(t)+8\mathrm{Li}_4(y)-4S_{2,2}(t)-4S_{2,2}(y)+2S_{1,3}(t)+2S_{1,3}(y)-4\mathrm{Li}_{1,3}\left(\frac{t}{y},y\right)-4\mathrm{Li}_{1,3}\left(\frac{y}{t},t\right)\right. \\
&+2\mathrm{Li}_{1,1,2}\left(1,\frac{t}{y},y\right)+2\mathrm{Li}_{1,1,2}\left(1,\frac{y}{t},t\right)+2\mathrm{Li}_{1,1,2}\left(\frac{t}{y},1,y\right)+2\mathrm{Li}_{1,1,2}\left(\frac{y}{t},1,t\right) \\
&\left.+2\mathrm{Li}_{1,1,2}\left(\frac{y}{t},\frac{t}{y},y\right)+2\mathrm{Li}_{1,1,2}\left(\frac{t}{y},\frac{y}{t},t\right)\right]+\mathcal{O}(\epsilon^5)\,. \qquad \text{(A.13)}
\end{aligned}
$$

The functions $\mathrm{Li}_n(t)$ are the usual polylogarithms and $S_{n,p}(t)$ are the Nielsen's generalized polylogarithms. We follow the conventions and definitions of Ref. [30] (see also Ref. [32]). The numerical evaluation of multiple polylogarithms has been automated in Ref. [33].

The results given above are sufficient to explicitly derive the one-loop soft-gluon current in the kinematics where one of the massive quarks is in the initial state. Such formal result is of interest, for example, in studies of the properties of massive gauge-theory amplitudes.

In the case of one non-zero mass, as needed for *Case 1*, the integral $M_3^{(SL)}$ reads:

$$
\begin{aligned}
&M_3^{(SL)}|_{m_j=0} = \Phi\,\frac{\pi^{-2+\epsilon}}{16}\frac{\Gamma(-\epsilon)\Gamma(1+2\epsilon)}{(p_i\cdot q)(p_j\cdot q)}\left(\frac{2\,p_i\cdot p_j}{(p_i\cdot q)(p_j\cdot q)}\right)^{\epsilon}\Bigg\{ \qquad \text{(A.14)} \\
&\frac{\Gamma(1+\epsilon)\Gamma(-2\epsilon)}{\Gamma(1-\epsilon)}\,{}_2F_1\left(-\epsilon,1+\epsilon,1-\epsilon;1-\frac{\alpha_i}{2}\right)-\frac{(-1-i\delta)^{-2\epsilon}}{\epsilon}\,{}_2F_1\left(-\epsilon,-\epsilon,1-\epsilon;1-\frac{\alpha_i}{2}\right)\Bigg\}\,.
\end{aligned}
$$

In the massless case, the result is simple and agrees with the one given in Ref. [9].

Finally, we remark that even in the case of equal masses, the result for the one-loop soft-gluon current is a function of two independent parameters, i.e. the case of two un-equal masses is not more complicated than the equal mass case. This is evident, for example, from Eq. (A.11).

Appendix B. Small-mass limit of $J_a^{\mu(1)}$

Following the methods of Ref. [15] one can independently derive the leading behavior of the one-loop soft-gluon current in the small-mass limit. Considering UV renormalized amplitudes, assuming all non-zero masses are equal and then taking the small-mass limit of both sides in Eq. (2), one easily derives:

$$
J_a(q;m\neq 0) = \sqrt{Z_{[g]}^{(m|0)}}\,J_a(q;m=0)+\mathcal{O}(m^2)\,. \qquad \text{(B.1)}
$$

In the above equation we take the strong coupling running with n_L+1 flavors, i.e. the heavy flavor is active. The factor $Z_{[g]}^{(m|0)}$ is given in the appendix of Ref. [15] through one loop and to all orders in ϵ. One can also check that upon decoupling the heavy flavor (in d-dimensions) the Z-factor in the above equation is exactly compensated through one loop, i.e. in the decoupling scheme, Eq. (B.1) simplifies to:

$$
J_a(q;m\neq 0) = J_a(q;m=0)+\mathcal{O}(m^2)\,. \qquad \text{(B.2)}
$$

The decoupling relations in d-dimension can be found, for example, in Ref. [34]; see also Section 4.

Appendix C. Pole structure of $J_a^{\mu(1)}$

One can provide an independent derivation of the poles of the UV renormalized, one-loop soft-gluon current from the known structure of the singularities of one-loop gauge theory amplitudes. Considering the amplitudes appearing in Eq. (2) as wide-angle scattering amplitudes, we can decompose them into jet, soft and hard functions respectively [12]:

$$\begin{aligned} M_a(n+1;q) &= I \times S_{ab} \cdot H_b \,, \\ M(n) &= i \times \sigma \cdot h \,. \end{aligned} \tag{C.1}$$

The jet functions I, i are diagonal in color. At one-loop they contain double and single poles. The soft functions S, σ are color matrices that have single poles only. The hard functions H, h are finite color vectors.

Applying the decomposition (C.1) to Eq. (2) and expanding each factor through one loop we get:

$$J_a^{(1)} = \left(I^{(1)} - i^{(1)} - \sigma^{(1)}\right) J_a^{(0)} + S_{ab}^{(1)} J_b^{(0)} + \left[\sigma^{(1)}, J_a^{(0)}\right] + \mathcal{O}(\epsilon^0)\,. \tag{C.2}$$

The separation of jet and soft functions is scheme dependent. We work in the formfactor scheme [13] where the jet function is a product of the square root of the formfactors corresponding to each external leg, and similarly in the massive case [15]. It then immediately follows that $I^{(1)} - i^{(1)} = f_g^{(1)}/2$, where $f_g^{(1)}$ is the one-loop correction to the UV-renormalized gluon form-factor [35, 36]:

$$f_g^{(1)} = \frac{\alpha_S}{2\pi}\left(-\frac{C_A}{\epsilon^2} - \frac{\beta_0}{\epsilon} + \mathcal{O}(\epsilon^0)\right)\,, \tag{C.3}$$

where $\beta_0 = 11C_A/6 - N_F/3$ and α_S is the $\overline{\mathrm{MS}}$ renormalized coupling at scale μ.

The explicit results for the soft functions read:

$$\begin{aligned} \sigma^{(1)} &= \frac{\alpha_S}{2\pi\epsilon}\,\frac{1}{2}\sum_{i\neq j=1}^{n} s_{ij} T_i \cdot T_j\,, \\ S_{ab}^{(1)} &= \sigma^{(1)}\delta_{ab} + \frac{\alpha_S}{2\pi\epsilon}\sum_{i=1}^{n} s_{gi}\,(T_g)_{ab}\cdot T_i\,, \end{aligned} \tag{C.4}$$

and the index g denotes the soft-gluon leg. The color matrices pertaining to the soft-gluon are $(T_g^a)_{cb} = i f_{cab}$. The functions $s_{ij} = s_{ji}$, not to be confused with partonic invariants, [3] depend on whether the legs i, j are both massive or not and can be found, for example, in Eq. (29) of Ref. [21].

Combining the above results and after some algebra we derive the following expression for the poles of the UV renormalized one-loop soft-gluon current:

$$J_a^{\mu(1)} = \frac{\alpha_S}{2\pi}\, i f^{abc}\sum_{i\neq j=1}^{n} T_i^b T_j^c\left(e_i^\mu - e_j^\mu\right)\left\{-\frac{1}{2\epsilon^2} - \frac{1}{2\epsilon}\left[\frac{\beta_0}{C_A} + \ln\left(\frac{-\mu^2(p_i\cdot p_j)}{2(p_i\cdot q)(p_j\cdot q)}\right) - h_{ij}\right] + \mathcal{O}(\epsilon^0)\right\}. \tag{C.5}$$

The function h_{ij} reads (see, for example, Eq. (29) of Ref. [21]):

$$h_{ij} = \ln\left(1+x^2\right) + \frac{2x^2}{1-x^2}\ln(x)\,, \quad \text{when} \;\; m_i \neq 0, m_j \neq 0\,, \tag{C.6}$$

and zero otherwise. The variable x is defined in Eq. (12). The function h_{ij} vanishes in the massless limit. In deriving the above result we have used color conservation $\sum_{i=1}^n T_i^a = 0$ and the identity $i f^{abc} T_i^a T_i^b = -(C_A/2) T_i^c$, or alternatively, $J_a^{\mu(0)} = \frac{i f^{abc}}{C_A}\sum_{i\neq j=1}^{n} T_i^b T_j^c\left(e_i^\mu - e_j^\mu\right)$.

[3] In this paper we do not use the notation s_{ij} to denote partonic invariants.

Eq. (C.5) agrees with Ref. [9]. To that end we need to convert the renormalized coupling to the bare one and recall the overall factor of g_S in Eq. (3). We also recall the discussion in Section 4 where we explain that we work with $n_f = n_l$ active flavors and that heavy quark loops in external gluon fields are exactly compensated by the decoupling relation. See also Refs. [37],[38], for more information on that point.

Eq. (C.5) applies to space-like kinematics as in *Case 1*. Continuation to any other kinematics is trivial; see Ref. [25] and Appendix D.

Appendix D. Analytical continuation to physical kinematics

It is often the case that calculations of scattering amplitudes are easier to perform in unphysical kinematics. Then the question arises how to analytically continue the result derived in such unphysical kinematics to the physical region. In this work, the continuation involves the momentum p_j, i.e. we need to continue results derived in kinematics where p_j is incoming to kinematics where p_j is outgoing.

The analytical continuation is more involved when p_j is massive. When the momentum p_j is incoming we have evaluated the soft current exactly in d-dimensions (see Eqns. (6, A.2, A.4, A.11, A.13, A.14). To continue the result to the timelike kinematics where p_j is outgoing one has to first express the result in a minimal number of variables. Given the scaling-invariance properties of the result, only two variables are truly independent. As such we take x and α_i, defined in Eqns. (9,12). The variables α_j and v can be eliminated through the relations $\alpha_i\alpha_j = 1 - v^2$ and $v = (1 - x^2)/(1 + x^2)$.

The rules for the analytical continuation in terms of the variables x and α_i (for general values of the masses $m_{i,j}$) are simple, see also Ref. [25]:

$$\begin{aligned} x &\to -x + i\delta\,, \\ \alpha_i &\to \alpha_i\,. \end{aligned} \tag{D.1}$$

The invariance of α_i is easy to understand, since $\alpha_i \sim (p_j \cdot q)/(p_i \cdot p_j)$. Not only is α_i invariant under $p_j \to -p_j$ [4] but, more importantly, its log is: $\ln(\alpha_i) = \ln((p_j \cdot q)) - \ln((p_i \cdot p_j)) + \ldots = \text{inv.}$ That is distinct from the case of $\alpha_j \sim 1/((p_j \cdot q)(p_i \cdot p_j))$ which itself is invariant but its log is not: $\ln(\alpha_j) = -\ln((p_j \cdot q)) - \ln((p_i \cdot p_j)) + \ldots \neq \text{inv.}$ That α_j must transform nontrivially also follows from the identity $\alpha_i\alpha_j = 1 - v^2$.

The practical implementation of the analytical continuation procedure requires the explicit extraction of all branching-point singularities around the point $x = 0$. Here is a typical example:

$$\mathrm{Li}_2\left(-\frac{1}{x^2}\right) = -\mathrm{Li}_2(-x^2) - \frac{\pi^2}{6} - 2\ln^2(x)\,, \tag{D.2}$$

and:

$$\ln(x^2) \to \ln(x^2) + 2i\pi \quad ; \quad \mathrm{Li}_n(1 - x^2) \to \mathrm{Li}_n(1 - x^2) - 2i\pi\frac{\ln^{n-1}(1 - x^2)}{(n-1)!}\,. \tag{D.3}$$

We have verified that with the help of Eq. (D.1) we can reproduce the first three orders in ϵ of the directly calculated integral $M_3^{(TL)}$, from the spacelike calculation of $M_3^{(SL)}$; see Appendix A. Starting from the fourth order in ϵ one would have to devise a similar procedure for the set of multiple polylogarithms that begin to appear, see Eq. (A.13). This presents an interesting direction for future work that can benefit from the number of recent applications of this class of functions in the context of gauge amplitudes in $\mathcal{N} = 4$ SYM theories [39].

When p_j is massless, the analytical continuation allows one to obtain the results for configuration *Case* 2 from the one for *Case* 1. For $m_j = 0$ the conformal variable vanishes, $x = 0$, which implies that the analytical continuation (D.1) becomes trivial. That can also be seen with a direct inspection of the integrals (A.2,A.3) and (A.7) in the case $m_j = 0$.

[4] Strictly speaking one inverts not the momentum p_j but the sign of all invarians $(k \cdot p_j)$ linear in p_j.

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